

MAS 117

Introduction to Algebra

Solutions 4

January 2007

1 What is important here is that each true property should be proved, while each false property should be provided with a counterexample.

(a) Reflexive, since a divides a . Not symmetric, since 2 divides 4 but 4 doesn't divide 2. Transitive since, if a divides b and b divides c , then $b = ma$ and $c = nb$, so $c = mna$, which means that a divides c .

(b) Not reflexive; 0 is not greater than 0. Not symmetric, since 2 is greater than 1 but 1 is not greater than 2. Transitive since if $a > b$ and $b > c$ then $a > c$.

(c) Reflexive since $|a| = |a|$. Symmetric since if $|a| = |b|$ then $|b| = |a|$. Transitive since if $|a| = |b|$ and $|b| = |c|$ then $|a| = |c|$.

(d) Not reflexive since $1 + 1$ is not a multiple of 5. Symmetric since $a + b = b + a$. Not transitive since $1 + 4$ and $4 + 6$ are multiples of 5 but $1 + 6$ is not.

(e) Reflexive? Does a country have a common border with itself? This is arguable either way. (You might like to think about Malta – does it even have a border?) Symmetric? yes, clearly. Not transitive since Spain and France share a border, as do France and Germany, but Spain and Germany do not.

2 The only one which is an equivalence relation is (c). The equivalence class of a complex number a consists of all complex numbers b with $|b| = |a|$. If $a = 0$, this is just the set $\{0\}$ with a single element 0; if $a \neq 0$, it is the set of points on a circle with centre at the origin and radius $|a|$ in the Argand diagram.

3 (a) Not a function since $\arg(0)$ is not defined.

(b) This function is not one-to-one, because $F(0) = F(1) = 0$. It is onto, because $F(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ while $F(x) \rightarrow \infty$ as $x \rightarrow \infty$, and so $F(x)$ takes any value between $-\infty$ and ∞ , that is, any real number. (A clear graph shows this.)

(c) This function is one-to-one since, if $e^x = e^y$ then (taking natural logarithms) we conclude that $x = y$. It is not onto, since it takes only positive values.

4 The argument begins “take $(x,y) \in R$ ”. If it happens that, for some x , there is no such pair in R , then the argument will not work.

For example, if R is the empty relation on a (nonempty) set, then R is symmetric and transitive but not reflexive. [Remember that “if A then B ” is true if A is false; so if there are no pairs $(x,y) \in R$, then “if $(x,y) \in R$ then $(y,x) \in R$ ” is true.]

5 The tables give all the functions in each case. (Each row of the table defines one function.)

$F(1)$	$F(2)$	$F(3)$	one-to-one?	onto?	$F(1)$	$F(2)$	one-to-one?	onto?
1	1	1	×	×	1	1	×	×
1	1	2	×	✓	1	2	✓	×
1	2	1	×	✓	1	3	✓	×
1	2	2	×	✓	2	1	✓	×
2	1	1	×	✓	2	2	×	×
2	1	2	×	✓	2	3	✓	×
2	2	1	×	✓	3	1	✓	×
2	2	2	×	×	3	2	✓	×
					3	3	×	×

So the answers are (a) 8, 0, 6; (b) 9, 6, 0.

Note that there cannot be a one-to-one function from a larger set to a smaller one, or an onto function from a smaller set to a larger one, explaining the zeros.